PROPERTIES OF FOURIER TRANSFORMS

- **Fourier Transform or Complex Fourier transform.**

Let f(x) be a function defined on (-∞,∞) and be piece wise Continuous in each finite partial interval and absolutely integrable in (-∞,∞), then

$$F\ \{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f\ (x)\ dx$$

Is called the Fourier Transform of f(x) and is denoted by F{f(x)} of $\overline{f}$(p)

The function f(x is called the inverse Fourier transform of $\overline{f}$(p)

$$F(x)=F^{-1}\{\overline{f}(p)\}.$$

- **Linearity property of Fourier Transform.**

If $\overline{f}$(p) and $\overline{g}$(p) are Fourier Transform of f(x) and g(x) respectively then

$$F\{af(x)+bg(x)\}=a\ \overline{f}(p)+h\ \overline{g}(p)$$

Where a and b are constants.

Proof : we have

$$F\{f(x)=\overline{f}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f\ (x)\ dx$$

$$F\{g(x)\}=\overline{g}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\ g(x)dx$$

$$F\{af(X)+bg(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}(af(x)+\ bg(x)\}\ dx$$

$$=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f\ (x)\ dx+\frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\ g(x)dx$$

$$=a\ \overline{f}(p)+b\ \ \overline{g}(p)$$

- **Change of Scale property.**

Theorem :1

If $\overline{f}$(p) is the complex Fourier transform of f(x),the Complex Fourier transform of f (ax) is 1/a $\overline{f}$(p/a)

Proof: we have

$\overline{f}$(p)= F{f(x)= $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f\,(x)\,dx$ ....(i)

New F{f(ax)= $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f\,(ax)\,dx$

=1/a. $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f\,(t)\,dt$

Putting ax=t

So that dx=1/a dt

=1/a $\overline{f}$(p/a), from(i)

Theorem :2 (for Fourier Sine Transform)

If $\overline{f_s}$(p is the Fourier Sine transform of f(x), then Fourier sine transform of

f(ax) is 1/a $\overline{f_s}$(p/a)

Proof: we have

$\overline{f}_s(p)$ =F$_s$ {f(x)}=(2/n) $\int_0^\infty f(x)\, sin\, px\, dx$ (i)

New F$_s$ {f(ax)}=(2/n) $\int_0^\infty f(ax)\, sin\, px\, dx$

Putting ax=1

So that dx=1/a dt, we have

F$_s$ {f(ax)}= 1/a.√(2/n) $\int_0^\infty f(t)$ sin $sin\, \frac{p}{a}\, t\, dt$

=1/a $\overline{f}_s$(p/a), from(i)

Theorem:3 (For Fourier Cosine Transform).

If $\overline{f}_c$(p) is the Fourier Cosine Transform of f(x), then Fourier cosine transform of f(ax) is 1/a $\overline{f}_c$(p/a)

If $\overline{f}$(p) is th complex Fourier transform of f(x), then complex Fourier transform $\overline{f}$(p) of f(x-a) is e^{ipa}

Proof : we have

$\overline{f}$(p)=F{f(x)}= $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f\,(x) dx$

Now F{f(x-a)}= $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f\,(x - a) dx$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx(a+t)} f(t)\, dt$$

Putting x-a=t

So that dx=dt

$$=e^{ipa}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f(t)\, dt$$

$=e^{ipa}\ \bar{f}(p)$ from (i)

□ **Modulation Theorem**

If $\bar{f}(p)$ is the Complex Fourier Transform of f(x) ,then the Fourier transform of f(x) cosax is $1/2[\bar{f}(p-a)+\bar{f}(p+a)]$

Proof: we have

$$\bar{f}(p)=F\{f(x)\}+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx$$

New $F\{f(x).\ \text{Cos}\ ax\}= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx\ \frac{e^{ipx}+e^{-ipx}}{2}$

$$=\frac{1}{2}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p-a)x} f(x)dx]$$

=1/2 [$\overline{f}$(p+a)+ $\overline{f}$(p-a)].

- If $\overline{f}$(p) and $\overline{f}_c$(p) are Fourier sine and cosine transform of f(x) respectively, then

(i) F_s\{f(x) cos ax)=1/2[$\overline{f}_s$(p+a)+ $\overline{f}_s$(p-a)]

(ii) F_s\{f(x) sin ax)=1/2[$\overline{f}_s$ (p+a)- $\overline{f}_s$(p-a)]

(iii) F_s\{f(x) sin ax)=1/2[$\overline{f}_s$(p-a)- $\overline{f}_s$(p+a)].

Proof: (i) we have

(i) F_s\{f(x) cos ax)=1/2[$\overline{f}_s$ (p+a)+ $\overline{f}_s$ (p-a)]

F_s\{f(x) cos ax)= $\frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(x)\, cosax.\, sin\, px\, dx$

$=\frac{1}{\sqrt{2\pi}}(1/2)$

$\int_0^{\infty} f(x)[\sin sin\,(p + a)\,x - \sin sin\,(p - a)x]\,dx$

$= \frac{1}{\sqrt{2\pi}}(1/2)[\int_0^{\infty} f(x)[\sin sin\,(p + a)\,xdx - \sin sin\,(p - a)x]\,dx\,]$

$$=\frac{1}{\sqrt{2\pi}} 1/2 \left[\int_0^\infty f(x)\, sin(p + a)x\, dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\, sin(p - a)x\, dx \right.$$

$$= ½ \left[\overline{f_s}(p+a) + \overline{f_s}(p-a) \right]..$$